Introduction and Overview

This book is about doing, understanding, and teaching mathematical problem solving. Most of the problems discussed are appropriate for college freshmen, with a tolerance of about 2 years; that is, they are generally accessible to tenth or eleventh graders, but mathematics majors at the junior level in college will often find them challenging. Few of the problems require a knowledge of mathematics as sophisticated as calculus in order to be solved. Virtually all, however, require a substantial amount of thinking. Understanding the nature of mathematical thinking is the issue at the core of this book, and pursuing it will lead us to a host of related issues. The following hypothetical experiment introduces some of the major ones.

Two groups of people participate in that experiment. The first group consists of a dozen mathematically talented undergraduates, say, the top 12 first-year mathematics majors at a particular university. The second group consists of a dozen members of the mathematics department at the same university. The mathematicians are randomly selected, save for one condition: They have not done any plane geometry for at least 10 years. (Surprisingly, this is not an unusual condition. Randomly selected mathematicians have not, in all likelihood, done any plane geometry since their high school days.) Each of the participants will be asked to solve a series of geometry problems similar to Problems 1.1 and 1.2, given below.

Problem 1.1 You are given two intersecting straight lines and a point P marked on one of them, as in the figure below. Show how to construct, using straightedge and compass, a circle that is tangent to both lines and that has the point P as its point of tangency to one of the lines.

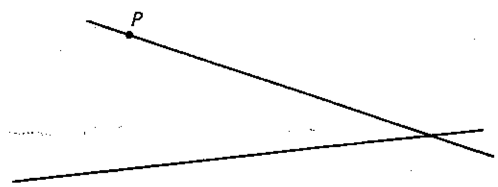

Figure 0.1

Problem 1.2 You are given a fixed triangle T with base B, as in the figure below. Show that it is always possible to construct, with straightedge and compass, a straight line that is parallel to B and that divides T into two parts of equal area. Can you similarly divide T into five parts of equal area?

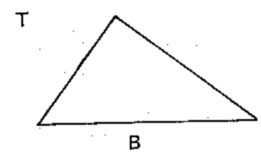

Figure 0.2

Problems like these, which are discussed extensively in the sequel, are "nonstandard" in that they are not typically covered in high school geometry courses. The college students are likely to possess more than enough factual knowledge to solve these problems. However, the nonstandard nature of the problems ensures that the students will not be able to solve them by simply recalling and applying familiar solution patterns (known as problem *schemata*). Nor will the problems be familiar to the faculty. Moreover, since none of the faculty has done geometry for many years, their initial recall of directly relevant facts and procedures is likely to be a good deal more shaky than the students'. We can expect both groups of subjects to find at least some of the problems challenging.

The subjects are trained to solve problems out loud. Their verbal reports as they work are recorded, and the recordings are transcribed. These transcripts (called *protocols*) and the written work produced by the subjects constitute our data. We consider two questions:

1. Which group will do better on the problems, and why?

depends heavily on a firm foundation of domain-specific resources. It is unrealistic to expect too much of these strategies.

The next section briefly describes the sense in which *problem* is used in this book, and describes my target group of students. The three subsequent sections examine, in detail and in order, the three points raised above.

What a Problem Is and Who the Students Are

The difficulty with defining the term *problem* is that problem solving is relative. The same tasks that call for significant efforts from some students may well be routine exercises for others, and answering them may just be a matter of recall for a given mathematician. Thus being a "problem" is not a property inherent in a mathematical task. Rather, it is a particular relationship between the individual and the task that makes the task a problem for that person. The word *problem* is used here in this relative sense, as a task that is difficult for the individual who is trying to solve it. Moreover, that difficulty should be an intellectual impasse rather than a computational one. (For example, inverting a 27×27 matrix would be an arduous task for me, and I would most likely make an arithmetic error in the process. Even so, inverting a given matrix is not a *problem* for me.) To state things more formally, if one has ready access to a solution schema for a mathematical task, that task is an exercise and not a problem.

Of the definitions in the *Oxford English Dictionary,* the one I prefer is the following: "**Problem.** A doubtful or difficult question; a matter of inquiry, discussion, or thought; a question that exercises the mind."

This is the sense in which Pólya uses the term and that establishes the context for his attempt to revive heuristic strategies as tools in problem solving. That is, heuristic strategies are techniques used by good problem solvers when they need to make progress on tasks that are problems for them. As Pólya put it, "The aim of heuristic is to study the methods and rules of discovery and invention" (1945, p. 112). Roughly characterized, heuristic strategies are techniques used by problem solvers when they run into difficulty. They are rules of thumb for making sense of, and progress on, difficult problems. The examples of heuristic usage given in Chapter 1 give some sense of how such strategies can be used. It should be clear from those examples that these techniques are subtle and difficult to use—especially since they are called into play precisely when the problem solver does not have a good idea of what to do next. Learning to use heuristics calls for a (reasonably) firm foundation of mathematical resources and for a fair amount of sophistication as well. For these reasons most of our discussions will be restricted to subjects

1. who are more or less adult problem solvers, say, college students or older,[1]
2. who have reasonable mathematical backgrounds, which preferably include some familiarity with the calculus,[2]
3. who are working on problems in the sense described above, and
4. who are truly making an effort to solve the problems that are (at least potentially) within their grasp.[3]

Note 1 There is a tolerance of perhaps 2 years at the lower level. As suggested above and as is discussed at length below, heuristics are subtle, complex, and highly abstract. As discussed in Chapter 1, these are only one component of a competent problem solver's arsenal of techniques. Control strategies are even more abstract; they are, in essence, operators whose domain is the space of heuristics. The reason for the age limit suggested here is simply to help ensure that the students will have some degree of mathematical maturity.

This is not to suggest that students should first be exposed to heuristics in college. The foundations for using such strategies can and should be established during the whole of a student's mathematical career. Indeed, if such groundwork was routinely done, much of the essentially remedial work I am compelled to do at the college level would be unnecessary. Younger students can recognize, appreciate, and mimic the use of such strategies. But anyone who thinks that fourth-graders, for example, can use these mathematical strategies in the way that Pólya described them either fails to understand the complexity of the strategies or fails to understand the lessons from Piaget's work.

Note 2 None of the mathematical tasks (I am reluctant to say "problems") used as examples in this chapter—and few of those used in my problem-solving courses, for that matter—require a knowledge of calculus for solution, but all do require some degree of sophistication and some fluency with basic mathematics. The calculus prerequisite is a simple way of assuring that the students have some degree of sophistication.

Note 3 The purpose of these constraints is to allow for the exploration of heuristic behavior in relatively favorable circumstances. The intent here is not to avoid a discussion of affective issues, for affective issues are clearly a major factor determining problem-solving behavior and success. Matters will be more straightforward, however, if that discussion is postponed for a while. A direct discussion of affective issues begins in Chapter 5.